APPENDIX C: PROJECTS

A useful learning tool is to complete the following two projects. If your lecturer chooses to use them you'll be assigned a number from 1 to 48 which corresponds to a group presentation from the following table. This will be your own personal group throughout both projects. Get to know it well. If for any reason you find that you don't get on with your group, let your lecturer know and you may be assigned a substitute.

If such projects are not assigned, it would still be useful for you to choose a group and carry out the projects on your own. At the very least you should study the sample project that is included at the end of this appendix.

For lecturers who decide to use these projects, solutions can be obtained from the author by emailing the author at christopherdonaldcooper@gmail.com. I can also supply a catalogue of all groups up to 100, with the exception of orders 64 and 72. This provides character tables, lists of subgroups etc.

Your task for Project 1 is to:
(1) Use the Todd-Coxeter algorithm to find the order of G (you may wish to simplify the presentation first, or split the group as a direct product if that is possible for your group).
(2) Construct the group table.
(3) Find the inverses of the elements.
(4) Find the orders of the elements.
(5) Construct the order profile.
(6) Find the centraliser of each element.
(7) Find the number of conjugates for each element.
(8) Find the class equation.

You task for Project 2 is to:
(1) Copy your group table from Project 1.
(2) Find the conjugacy classes.
(3) Find the normal subgroups.
(4) Find the centre.
(5) Find the derived subgroup.
(6) Draw the lattice of normal subgroups.
(7) Chose a minimal normal subgroup and construct the group table of the corresponding quotient group (code each coset by a representative).
(8) Choose a maximal subgroup, H, and induce the trivial character of H up to G. Calculate the inner product of this character with itself and hence decompose it into linear characters. (This may help you in (9), or you may want to complete the character table first.)
(9) Construct the character table for G;

(10) (Optional) Any other information about G that you wish to include.

Each project is to come in three sections:

TITLE PAGE in which you give your name, your student number, the number (1-48) of your presentation together with the presentation itself;

PART A in which you list your answers;

PART B in which you show the details of your calculations.

The intention is for all the calculations to be performed by hand. If you are an efficient programmer with lots of time and enthusiasm you might like write a program to simulate hand-implementation, though it is not recommended. If you do so you should include a fully documented listing of your program in an appendix and your output should show complete working *as if* it had been done by hand, not just final answers. Also, whether or not you implement by hand or by computer, you should be in a position to explain the details of any step if called on to do so.

You should consider carrying out possible simplifying 'pre-processing', if you think it might simplify your calculations, provided you use the Todd-Coxeter algorithm somewhere. For example you may recognise that your group is a direct product of smaller groups, or you may choose to rewrite the presentation in an equivalent, but simpler, form.

You are invited to check with your lecturer at various stages to see if it looks alright so far. This will avoid the problem of making an early mistake and wasting a lot of time. Also if the Todd-Coxeter process doesn't seem to be terminating don't just continue mindlessly. It may mean that you aren't using the algorithm correctly, or it could mean that your choice strategy is not as efficient as it could be.

If you get a contradiction it probably means you have made an error. You need to work carefully and neatly. A blank table is provided in this appendix. However it is possible that you obtain a contradiction even though you haven't made an error. This is rare, but it could mean that you have made poor choices in the algorithm.

The Todd-Coxeter algorithm can be tedious but you really learn to understand it by carrying it out on a reasonable-size group. Be warned that it can take several hours and working systematically and carefully can avoid having to start again. But remember that in many cases the presentation can be simplified in a fairly obvious way, or can be decomposed into direct products. These can greatly decrease the amount of work.

A sample project follows the list of presentations in this appendix.

YOUR GROUP IS ONE OF THE FOLLOWING

#	presentation
1	$\langle A,B \mid A^4, B^4, (AB)^2, (A^{-1}B)^2 \rangle$
2	$\langle A,B \mid A^8, B^2, AB = BA^3 \rangle$
3	$\langle A,B, C, D \mid A^2, B^2, C^3, D^2, CA = ADC, CD = AC \rangle$
4	$\langle A,B,C \mid A^4, B^2, C^2, AB = BA^{-1}, AC = CA, BC = CB \rangle$
5	$\langle A,B,C \mid A^2, B^4, C^2 = B^2, BCB = C, AB = BA, AC = CA \rangle$
6	$\langle A,B, C \mid A^3, B^3, C^2, CA = A^{-1}C \rangle$
7	$\langle A,B \mid A^4, B^2, (A^{-1}B)^4, (A^2B)^2 \rangle$
8	$\langle A,B, C, D \mid A^2, B^2, CA = ABC, CB = AC \rangle$
9	$\langle A,B, C \mid A^3, B^2, C^3, (AB)^2 \rangle$
10	$\langle A,B \mid A^2, B^8, AB^3 = BA \rangle$
11	$\langle A,B,C \mid A^2, B^2, C^2, A^{-1}BCA = BC, CAB = BCA \rangle$
12	$\langle A,B, C, D \mid A^2, B^2, CA = ABC, CB = AC, CACB = ABCAC \rangle$
13	$\langle A,B, C \mid A^5, B^2, C^2, BA = A^{-1}B \rangle$
14	$\langle A,B,C \mid A^2 = C^2, B^2, C^4, CAC = A, AB = BA, BC = CB \rangle$
15	$\langle A,B \mid A^4, B^4, AB = BA^{-1} \rangle$
16	$\langle A,B,C \mid A^3, B^3, C^4, C^{10}, ABA = B \rangle$

17	$\langle A,B \mid A^8, A^4 = B^2, AB = BA^{-1} \rangle$
18	$\langle A,B \mid A^8, B^2, AB = BA^{-1} \rangle$
19	$\langle A,B \mid A^8, B^2, AB = BA^{-3} \rangle$
20	$\langle A,B, C, D \mid A^2, B^2, C^3, D^2, CA = ABC,$ $CB = AC \rangle$
21	$\langle A,B,C \mid A^4, B^2 = A^2, C^2, AB = BA^{-1}, AC = CA,$ $BC = CB \rangle$
22	$\langle A,B \mid A^4, B^4, BAB = A \rangle$
23	$\langle A,B \mid A^4, B^4, BA = AB^3 \rangle$
24	$\langle A,B \mid A^2, B^8, A^{-1}BAB^{-1} = B^4 \rangle$
25	$\langle A,B \mid A^8, B^2, (AB)^2 = B^2 \rangle$
26	$\langle A,B,C \mid A^3, B^4, C^2, BA = A^{-1}B, CB = B^{-1}C \rangle$
27	$\langle A,B,C \mid A^2, B^2, C^2, (ABC)^2 = (AB)^2,$ $ABC = BCA \rangle$
28	$\langle A,B \mid A^2 = B^4, B^8, AB^{-1} = BA \rangle$
29	$\langle A,B,C \mid A^2, B^2, C^2, ABC = BCA = CAB \rangle$
30	$\langle A,B \mid A^8, B^2, BA^3B = A \rangle$
31	$\langle A,B,C \mid A^2, B^4, C^2, BCB = C, AB = BA,$ $AC = CA \rangle$
32	$\langle A,B \mid A^4, B^4, (AB)^2, (AB^{-1})^2 \rangle$
33	$\langle A,B, C \mid A^3 = C, B^2 , C^4, (AB)^2 \rangle$
34	$\langle A,B \mid A^2, B^8, AB^3A = B \rangle$
35	$\langle A,B \mid A^4, B^4, (AB)^2, ABA^2B^{-1}A \rangle$
36	$\langle A,B \mid A^8, A^4 = B^2, BAB = A^3 \rangle$

37	$\langle A,B \mid A^8, B^2, (A^{-1}B)^2 \rangle$
38	$\langle A,B,C \mid A^4, B^3, C^2, AB = B^{-1}A, ACA = C \rangle$
39	$\langle A,B \mid A^2, B^8, B^{-1}A^{-1}BA = B^2 \rangle$
40	$\langle A,B,C \mid A^4, B^2 = A^2, C^2, B^{-1}AB = A^3, AC = CA, BC = CB \rangle$
41	$\langle A,B \mid A^8, B^2, AB = BA^5 \rangle$
42	$\langle A,B \mid A^4, B^4, (AB)^2 = B^2 \rangle$
43	$\langle A,B, C \mid A^2 = C, B^2, C^6, (AB)^2 \rangle$
44	$\langle A,B,C \mid A^2, B^2, C^4, CAC = A, AB = BA, BC = CB \rangle$
45	$\langle A,B \mid A^8, B^2, (AB)^2 \rangle$
46	$\langle A,B,C \mid A^2, B^2, C^2, ACB = CBA = BAC \rangle$
47	$\langle A,B,C \mid A^4, B^2, C^2, B^{-1}AB = A^3, AC = CA, BC = CB \rangle$
48	$\langle A,B \mid A^8, A^4 = B^2, A^4 = (AB)^2 \rangle$

SAMPLE PROJECT 1

TITLE PAGE:

A. Student
Student Number: 123456
Group Number: 0
Presentation:
$G = \langle A, B, C \mid A^3 B^2, C^6, C^4, AB^{-1}AB, AC = CA, BC = CB \rangle$

PART A:

(1) Group Order: $|G| = 12$

(2) Group Table:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	5	6	7	1	8	10	3	11	4	12	9
3	3	8	1	9	6	5	12	2	4	11	10	7
4	4	10	9	1	7	12	5	11	3	2	8	6
5	5	1	8	10	2	3	4	6	12	7	9	11
6	6	3	2	11	8	1	9	5	7	12	4	10
7	7	4	11	2	10	9	1	12	6	5	3	8
8	8	6	5	12	3	2	11	1	10	9	7	4
9	9	11	4	3	12	7	6	10	1	8	2	5
10	10	7	12	5	4	11	2	9	8	1	6	3
11	11	12	7	6	9	10	8	4	2	3	5	1
12	12	9	10	8	11	4	3	7	5	6	1	2

(3) Inverses:

x	1	2	3	4	5	6	7	8	9	10	11	12
x^{-1}	1	5	3	4	2	6	7	8	9	10	12	11

(4) Orders of Elements:

x	1	2	3	4	5	6	7	8	9	10	11	12
$\|x\|$	1	3	2	2	3	2	2	2	2	2	6	6

(5) Order Profile:

order	1	2	3	4	6
number	1	7	2	0	2

(6) Centralisers:
C(1) = G;
C(2) = C(5) = {1, 2, 5, 9, 11, 12};
C(3) = {1, 3, 4, 9};
C(4) = {1, 3, 4, 9};
C(6) = {1, 6, 7, 9};
C(7) = {1, 6, 7, 9};
C(8) = {1, 8, 9, 10};
C(9) = G;
C(10) = {1, 8, 9, 10};
C(11) = C(12) = {1, 2, 5, 9, 11, 12}

(7) The number of conjugates:

x	1	2	3	4	5	6	7	8	9	10	11	12
#conj	1	2	3	3	2	3	3	3	1	3	2	2

(8) Class equation: 12 = 1∗2 + 2∗2 + 3∗2

PART B:

The given presentation was

G = ⟨A, B, C|A^3, B^2, C^6, C^4, $AB^{-1}AB$, AC=CA, BC=CB⟩.

This can be simplified to

G = ⟨A, B, C|A^3 B^2, C^2, $AB^{-1}AB$, AC=CA, BC=CB⟩

because, if $C^6 = C^4 = 1$ then $C^2 = 1$. Also the relations can be rewritten as relators:

G = ⟨A, B, C|A^3 B^2, C^2,$AB^{-1}AB$, $ACA^{-1}C^{-1}$, $BCB^{-1}C^{-1}$⟩.

Finally, since $B^2 = C^2 = 1$ we may write B^{-1} as B and C^{-1} as C to obtain

G = ⟨A, B, C|A^3 B^2, C^2,ABAB, ACAC, BCBC⟩.

	A	A	A	B	B	C	C	A	B	A	B
1	2	5	**1**	3	**1**	4	**1**	2	6	3	**1**
2	5	1	**2**	6	**2**	7	**2**	5	8	6	**2**
3	8	6	**3**	1	**3**	9	**3**	8	5	1	**3**
4	10	7	**4**	9	**4**	1	**4**	10	12	9	**4**
5	1	2	**5**	8	**5**	10	**5**	1	3	8	**5**
6	3	8	**6**	2	**6**	11	**6**	3	1	2	**6**
7	4	10	**7**	11	**7**	2	**7**	4	9	11	**7**
8	6	3	**8**	5	**8**	12	**8**	6	2	5	**8**
9	11	12	**69**	4	**9**	3	**9**	11	7	4	**9**
10	7	4	**10**	12	**10**	5	**10**	7	11	12	**10**
11	12	9	**11**	7	**11**	6	**11**	12	10	7	**11**
12	9	11	**12**	10	**12**	8	**12**	9	4	10	**12**

	A	C	A	C	B	C	B	C
1	2	7	4	**1**	3	9	4	**1**
2	5	10	7	**2**	6	11	7	**2**
3	8	12	9	**3**	1	4	9	**3**
4	10	5	1	**4**	9	3	1	**4**
5	1	4	10	**5**	8	12	10	**5**
6	3	9	3	**6**	2	7	11	**6**
7	4	1	2	**7**	11	6	2	**7**
8	6	11	12	**8**	5	10	12	**8**
9	11	6	3	**9**	4	3	1	**9**
10	7	2	5	**10**	12	8	5	**10**
11	12	8	6	**11**	7	2	6	**11**
12	9	3	8	**12**	10	5	8	**12**

	A	**B**	**C**
1	(2)	(3)	(4)
2	(5)	(6)	(7)
3	(8)	1	(9)
4	(10)	9	1
5	1	8	10
6	3	2	(11)
7	4	11	2
8	6	5	(12)
9	11	4	3
10	7	12	5
11	12	7	6
12	9	10	8

The group table is:

		A	B	C	2A	2B	2C	3A	3C	4A	6C	8C
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	5	6	7	1	8	10	3	11	4	12	9
3	3	8	1	9	6	5	12	2	4	11	10	7
4	4	10	9	1	7	12	5	11	3	2	8	6
5	5	1	8	10	2	3	4	6	12	7	9	11
6	6	3	2	11	8	1	9	5	7	12	4	10
7	7	4	11	2	10	9	1	12	6	5	3	8
8	8	6	5	12	3	2	11	1	10	9	7	4
9	9	11	4	3	12	7	6	10	1	8	2	5
10	10	7	12	5	4	11	2	9	8	1	6	3
11	11	12	7	6	9	10	8	4	2	3	5	1
12	12	9	10	8	11	4	3	7	5	6	1	2

The inverses are found by looking for the positions of the 1’s in the group table.

The centralisers are obtained by scanning along a row and down the corresponding column, noting where we get a match. Because elements and their inverses have the same centralisers we can skip some elements.

The number of conjugates is the index of the centraliser. So, for example, 6 has 12/4 = 3 conjugates because its centraliser has order 4.

Note that there are 4 elements with 2 conjugates so they must be in 2 separate classes.

Similarly the number of classes of size 3 is 6/3 = 2.

SAMPLE PROJECT 2

TITLE PAGE:
A. Student
Student Number: 123456
Group Number: 0
Presentation: G = $\langle A,B,C | A^3 B^2, C^6, C^4, AB^{-1}AB, AC=CA, BC=CB \rangle$

PART A:
(1) Group table: copied from Project 1.

	1 **1**	A **2**	B **3**	C **4**	A^2 **5**	AB **6**
1	1	2	3	4	5	6
2	2	5	6	7	1	8
3	3	8	1	9	6	5
4	4	10	9	1	7	12
5	5	1	8	10	2	3
6	6	3	2	11	8	1
7	7	4	11	2	10	9
8	8	6	5	12	3	2
9	9	11	4	3	12	7
10	10	7	12	5	4	11
11	11	12	7	6	9	10
12	12	9	10	8	11	4

	AC 7	BA 8	BC 9	CA 10	ABC 11	BAC 12
1	7	8	9	10	11	12
2	10	3	11	4	12	9
3	12	2	4	11	10	7
4	5	11	3	2	8	6
5	4	6	12	7	9	11
6	9	5	7	12	4	10
7	1	12	6	5	3	8
8	11	1	10	9	7	4
9	6	10	1	8	2	5
10	2	9	8	1	6	3
11	8	4	2	3	5	1
12	3	7	5	6	1	2

(2) Conjugacy Classes:

Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
1	2 5	3 6 8	9	11 12	4 7 10

(3) Normal subgroups:

$G_0 = \{1\} = \Gamma_1$;

$G_1 = \{1, 9\} = \Gamma_1 + \Gamma_4$;

$G_2 = \{1, 2, 5\} = \Gamma_1 + \Gamma_2$;

$G_3 = \{1, 2, 5, 9, 11, 12\} = \Gamma_1 + \Gamma_2 + \Gamma_4 + \Gamma_5$;

$G_4 = \{1, 2, 3, 5, 6, 8\} = \Gamma_1 + \Gamma_2 + \Gamma_3$;

$G_5 = \{1, 2, 4, 5, 7, 10\} = \Gamma_1 + \Gamma_2 + \Gamma_6$ and

G itself.

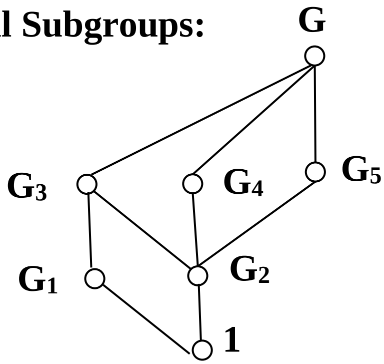
(4) Centre:

$Z(G) = G_1 = \Gamma_1 + \Gamma_4$.

(5) Derived Subgroup:

$G' = G_2 = \Gamma_1 + \Gamma_2$.

(6) Lattice of Normal Subgroups:



(7) Quotient Groups:

G/G_1:

	1	**2**	**3**	**5**	**6**	**8**
1	1	2	3	5	6	8
2	2	5	6	1	8	3
3	3	6	1	6	5	2
5	5	1	6	2	3	6
6	6	8	5	3	1	5
8	8	3	2	6	5	1

(8) Inducing the trivial character of a maximal subgroup:

G_3 = {1, 2, 5, 9, 11, 12} is a maximal subgroup.

Inducing the trivial character of G_3 up to G we get the character θ for G:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
size	**1**	**2**	**3**	**1**	**2**	**3**
$\chi 1$	1	1	1	1	1	1
θ	2	2	0	2	2	0
orders	**1**	**3**	**2**	**2**	**6**	**2**

$\langle\theta \mid \theta\rangle = 2$.

$\theta = \chi_1$ + another linear character.

(9) Character Table:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
size	**1**	**2**	**3**	**1**	**2**	**3**
χ_1	1	1	1	1	1	1
χ_2	1	1	–1	1	1	–1
χ_3	2	–1	0	2	–1	0
χ_4	1	1	1	–1	–1	–1
χ_5	1	1	–1	–1	–1	1
χ_6	2	–1	0	–2	1	0
order	**1**	**3**	**2**	**2**	**6**	**2**

(10) Other information:

Sylow Subgroups:

***p* = 2: {1, 3, 4, 9}, {1, 6, 7, 9} and {1, 8, 9, 10}, all isomorphic to V_4.**

***p* = 3: {1, 2, 5}, which is isomorphic to C_3.**

G is soluble but not nilpotent.

PART B:

Conjugacy classes:

Clearly 1 and 9 are classes of size 1. $3^{-1}.2.3 = 3.2.3 = 8.3 = 5$. Since 2 and 5 only have 2 conjugates this must be all. This leaves 11 and 12 as the remaining class of size 2. Conjugating 3 by 2 gives 6 and conjugating 6 by 2 gives 8. There are no further conjugates so {3, 6, 8} is a conjugacy class, leaving {4, 7, 10} as the remaining one.

Normal subgroups:

Of course there is the trivial subgroup and the whole group.

Normal subgroups of size 2: Such a group must be made up of two classes of size 1 and clearly {1, 9} is the only possibility.

Normal subgroups of size 3: Such a group must be cyclic and so the non-trivial elements must have order 3. So there is only one such subgroup: {1, 2, 5}.

Normal subgroups of size 4: Since there are no elements of order 4, such a subgroup would have to be made up of the identity and 3 elements of order 2. But the onlu conjugacy classes consisting of elements of order 2 are {9}, {3, 6, 8} and {4, 7, 10}. The only possibilities would be {1, 3, 6, 8} and {1, 4, 7, 10} but neither of these is closed. So there are no normal subgroups of order 4.

Normal subgroups of order 6: Well, any subgroup of order 6 would be normal, having index 2. The only subgroups of order 6 are C_6 and D_6. The only cyclic subgroup of order 6 is $\langle 11 \rangle = \{1, 11, 5, 9, 2, 12\}$.

A normal subgroup that is isomorphic to D_6 would have 2 elements of order 3 and 3 elements of order 2. The only possibilities are {1, 2, 5, 3, 6, 8} and {1, 2, 5, 4, 7, 10}.

Another argument is that if H = {1, 2, 5} then H is normal and G/H has order 4. This can't be cyclic because there is no element of G whose order is divisible by 4, So $G/H \cong \mathbf{V}_4$. This has three subgroups K/H of order 2 and such subgroups would have order 6. So we know that G has 3 subgroups of order 6, and we have found them.

Centre:

The centre consists of all the elements in a conjugacy class of size 1. So $Z(G) = G_1 = \Gamma_1 + \Gamma_4$.

Derived Subgroup:

G/G_2 has order 4 and so is abelian. Hence $G' \leq G_2$. But since G_2 has order 3, if $G' < G$ then $G' = 1$ and so G is abelian, which clearly it isn't. Hence $G' = G_2 = \Gamma_1 + \Gamma_2$.

Quotient Groups:

There are two minimal normal subgroups: G_1 and G_2.

The cosets (left and right) of G_1 in G are:

1 9	2 11	3 4	5 12	6 7	8 10

The group table for G/G_1 (representing each coset by a representative) is:

	1	**2**	**3**	**5**	**6**	**8**
1	1	2	3	5	6	8
2	2	5	6	1	8	3
3	3	6	1	6	5	2
5	5	1	6	2	3	6
6	6	8	5	3	1	5
8	8	3	2	6	5	1

This is abelian and so must be $\mathbf{C}_6$ rather than $\mathbf{D}_6$.

The cosets (left and right) of G_2 in G are:

1 2 5	3 6 8	4 7 10	9 11 12

The group table for G/G_1 (representing each coset by a representative) is:

	1	**3**	**4**	**9**
1	1	3	4	9
3	3	1	9	4
4	4	9	1	3
9	9	4	3	1

This is abelian. By examining the diagonal we can see that it must be $\mathbf{V}_4$ rather than $\mathbf{C}_4$.

WARNING: Here the group table for the quotient could be found directly from the group table for G. This will not always be the case. If the product of two representatives does not happen to be a representative, one should replace the product by the representative of the coset in which that product occurs.

Inducing the trivial character of a maximal subgroup:

G_3 = {1, 2, 5, 9, 11, 12} is a maximal subgroup. Inducing the trivial character of G_3 up to G we get the character θ for G:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
size	**1**	**2**	**3**	**1**	**2**	**3**
χ_1	1	1	1	1	1	1
θ	2	2	0	2	2	0
order	**1**	**3**	**2**	**2**	**6**	**2**

The value an induced character is the index of G_3 in G (which is 2) times the proportion of the conjugacy classes that lie in in G_3 (either 0 or 1 since G_3 is a normal subgroup and so each class is either all in or none in) times the average value of the character (which is 1).

$\langle\theta \mid \theta\rangle = \frac{1}{12}(4.1 + 4.2 + 0.3 + 4.1 + 4.2 + 0.3) = 2.$

Hence θ is not irreducible. But 2 is a sum of squares only as $1^2 + 1^2$ so θ is the sum of 2 different linear characters.

$\langle\chi_1 \mid \theta\rangle = \frac{1}{12}(1.2.1 + 1.2.2 + 1.0.3 + 1.2.1 + 1.2.2 +$

$1.0.3) = 1$ so one of these linear characters is χ_1. (Of course once we saw that $\langle\chi_1 \mid \theta\rangle$ had to be positive we could see that.)

Hence $\theta - \chi_1$ must be an irreducible character and so we obtain a second row of the character table:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
size	**1**	**2**	**3**	**1**	**2**	**3**
$\chi 1$	1	1	1	1	1	1
θ	2	2	0	2	2	0
$\theta - \chi_1$	1	1	–1	1	1	–1
orders	**1**	**3**	**2**	**2**	**6**	**2**

Character Table:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
size	**1**	**2**	**3**	**1**	**2**	**3**
χ_1	1	1	1	1	1	1
χ_2	1	1	–1	1	1	–1
χ_3	2	–1	0	2	–1	0
χ_4	1	1	1	–1	–1	–1
χ_5	1	1	–1	–1	–1	1
χ_6	2	–1	0	–2	1	0
order	1	3	2	2	6	2

The conjugacy classes of $H = G/G_1$ are {1}, {2, 5}, {3, 6, 8}. This can be done in the usual way but noting that G/G_1 has order 6 and is not cyclic (there are no elements of order 6), it must be $\mathbf{D}_6$. This has 3 conjugacy classes, namely the identity, the 2 elements of order 3 and the 3 elements of order 2.

The character table of H is thus:

class	1	2	3
size	**1**	**2**	**3**
χ_1	1	1	1
χ_2	1	1	–1
χ_3	2	–1	0
order	**1**	**3**	**2**

NOTES:

χ_1 is the trivial representation.

χ_2 was obtained by inducing from H/K where K is the normal subgroup of order 3.

χ_3 was obtained by orthogonality.

Inducing up from H to G we get 3 irreducible characters table for G:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
size	**1**	**2**	**3**	**1**	**2**	**3**
χ_1	1	1	1	1	1	1
χ_2	1	1	–1	1	1	–1
χ_3	2	–1	0	2	–1	0
χ_4						
χ_5						
χ_6						
order	**1**	**3**	**2**	**2**	**6**	**2**

The remaining degrees must satisfy $n_4^2 + n_5^2 + n_6^2 = 6$ so we may take $n_4 = n_5 = 1$ and $n_6 = 2$.

Since each class is its own inverse the entries in the character table are all real.
Since the elements of Γ_3 and Γ_6 have order 2, the entries in those columns must be ±1 for the degree 1 representations and ±2 or 0 for the degree 2.
Also, the entries χ_{42} and χ_{52} are real cube roots of unity and so must be both 1.

Thus we may complete these columns as follows:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
size	**1**	**2**	**3**	**1**	**2**	**3**
χ_1	1	1	1	1	1	1
χ_2	1	1	–1	1	1	–1
χ_3	2	–1	0	2	–1	0
χ_4	1	1	1	a	b	–1
χ_5	1	1	–1	c	d	1
χ_6	2	x	0	e	f	0
order	**1**	**3**	**2**	**2**	**6**	**2**

NOTE: The possibilities 1, 1, –1 for the last three entries in the 3rd and 6th columns violate the condition on the sum of squares of the entries down these columns.

By orthogonality between the 1st and 2nd columns get
$x = -1$.
By orthogonality between the 1st and 4th and 3rd and 4th rows we get $a + b = -2$ and $2a - b = -1$
From which we conclude that $a = b = -1$.

By orthogonality between the 5th and 3rd rows we get:
$2 – 2 + 0 + 2c – 2d + 0 = 0$ and hence $c = d$.

By orthogonality between the 1st row and the 5th we get:
$1 + 2 – 3 + c + 2d + 3 = 0$ and hence, since $c = d$, they are both equal to – 1.

By orthogonality between the 4th and 5th columns we get $e = – 2$ and $f = 1$.

This sounds rather ad hoc. We would have been better to use an extra quotient group.
G/G_1 is isomorphic to $\mathbf{V}_4$ which has character table:

	$\mathbf{\Gamma_1}$	$\mathbf{\Gamma_4}$	$\mathbf{\Gamma_5}$	$\mathbf{\Gamma_6}$
size	**1**	**1**	**2**	**3**
	1	1	1	1
	1	1	–1	–1
	1	–1	1	–1
	1	–1	–1	1
order	**1**	**2**	**2**	**2**

The cosets in G/G_1 are:

1 2 5	3 6 8	4 7 10	9 11 12

So inducing up to G we get the irreducible characters:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
elements	**1**	**2 5**	**3 6 8**	**9**	**11 12**	**4 7 10**
size	**1**	**2**	**3**	**1**	**2**	**3**
	1	1	1	1	1	1
	1	1	1	–1	–1	–1
	1	1	–1	–1	–1	1
	1	1	–1	1	1	–1
orders	**1**	**3**	**2**	**2**	**6**	**2**

Two of these we already had, but we get 2 new ones. Adding them to our character table we get:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
size	**1**	**2**	**3**	**1**	**2**	**3**
χ_1	1	1	1	1	1	1
χ_2	1	1	–1	1	1	–1
χ_3	2	–1	0	2	–1	0
χ_4	1	1	1	–1	–1	–1
χ_5	1	1	–1	–1	–1	1
χ_6						
order	**1**	**3**	**2**	**2**	**6**	**2**

Clearly χ_6 must have degree 2 for the sum of squares of the degrees to add up to 12. The rest of the table can be completed by column orthogonality.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
size	**1**	**2**	**3**	**1**	**2**	**3**
χ_1	1	1	1	1	1	1
χ_2	1	1	–1	1	1	–1
χ_3	2	–1	0	2	–1	0
χ_4	1	1	1	–1	–1	–1
χ_5	1	1	–1	–1	–1	1
χ_6	2	–1	0	–2	1	0

Note that the character we got from inducing up from a subgroup is one we could have got by inducing up from the quotient group. Inducing up from quotient groups is far better than using subgroups. But when you have a group with very few normal subgroups the subgroup inducing might be all we have. Ad hoc arguments like we used at first are OK but can get very messy. The moral is to use quotient group induction first, and only fall back on the other methods as a last resort.

Sylow subgroups:

A Sylow 2-subgroup would have order 4. It couldn't be cyclic as there are no elements of order 4. So they would be generated by 2 commuting elements of order 2.

The elements of order 2 are: 3, 4, 6, 7, 8, 9, 10. To find commuting pairs we look at the centralisers. But hey, the centralisers provide us with 3 ready-made subgroups of order 4:

{1, 3, 4, 9}, {1, 6, 7, 9} and {1, 8, 9, 10}. Are there any more?

The number of Sylow 2-subgroups is congruent to 1 modulo 2 and divides 12. The only such numbers are 1, 3 and we already have 3 Sylow 2-subgroups, so these are all there are.

A Sylow 3-subgroup must have order 3. We have the normal subgroup {1, 2, 5}. Any more? No, because all Sylow subgroups, for a particular prime, can be obtained from one another by conjugation. Once one is normal there can be no others.

Hence the Sylow *p*-subgroups are:
***p* = 2: {1, 3, 4, 9}, {1, 6, 7, 9} and {1, 8, 9, 10}, all isomorphic to V_4.**
***p* = 3: {1, 2, 5}, which is isomorphic to C_3.**

G is soluble:
G′ has order 3 and hence is cyclic. Thus G″ = 1.